Correspondence

On Longitudinal Waves in a Hot-Nonuniform Plasma

Recently, measurements have been made of wave propagation whose characteristic velocities have been on the order of the speed of sound for the electron gas, $(\gamma kT/m)^{1/2}$ where γ is a constant on the order of unity [1], [2]. Harmonic generation of these modes has been detected [3], [4] and they have been used to study drift velocities in the positive column of a mercury vapor discharge tube [5]. In this correspondence, we will study the propagation of these temperature dependent waves in a slightly inhomogeneous hot plasma. The positive column of a mercury vapor discharge possesses this slight inhomogeneity in its "long" dimension [6].

The equation of motion for electrons in an electron gas at rest is

$$Nm\frac{d\bar{v}}{dt} = -Ne[\bar{E} + \bar{v}\times\bar{B}] - \text{grad}\, P \quad (1)$$

where N = the electron density, $P=\gamma NkT_e$, T_e = electron temperature and γ (the compression constant) is on the order of unity [7].

With a time dependence of the form $\epsilon^{i\omega t}$, (1) combined with Maxwell's equations, and the definitions for conduction current and charge density, one can obtain the wave equation for a wave in a plasma. By linearizing, one obtains (see Appendix)

$$\text{curl curl}\,\bar{E} - \frac{\omega^2}{c^2}\epsilon_p\bar{E} - \alpha^2\,\text{grad div}\,\bar{E} + \frac{\alpha^2}{N_0}\text{grad}\,N_0\,\text{div}\,\bar{E} = 0 \quad (2)$$

where

$$\epsilon_p = \left(1 - \frac{\omega_p^2}{\omega^2}\right), \qquad \alpha^2 = \frac{\gamma kT_e}{m_eC^2}$$

and N_0 is the steady-state electron density. It has been assumed that T_e is a constant even though isothermal conditions do not exist. For some idealized cases, this can be be effected by a suitable choice of γ.

Equation (2) is the general wave equation for an inhomogeneous hot electron plasma. The first two terms correspond to the usual wave equation with a space-dependent dielectric constant; the third term gives the effect of the electron temperature, and the last term originates from an electric field caused by a static density gradient [8], [9].

In an infinite homogeneous medium, it is not difficult to show that with plane waves propagating in the z direction with a z dependence of the form $\epsilon^{-i\beta z}$, that a resulting dispersion relation is of the form

$$\left(\epsilon_p - \frac{c^2\beta^2}{\omega^2}\alpha^2\right)\left(\epsilon_p - \frac{c^2\beta^2}{\omega^2}\right)^2 = 0 \quad (3)$$

which yields a longitudinal mode when above the plasma frequency with a phase velocity

$$\frac{\omega}{\beta} = v_{ph} \sim \sqrt{\frac{\gamma kT_e}{m_e}}. \quad (4)$$

At this point, we are going to direct our attention to this longitudinal or hot plasma mode only and assume it to be excited and propagating in the direction of the inhomogeneity of the plasma. Equation (2) can then be written as

$$\ddot{E}_z + \frac{k_0^2}{\alpha^2}E_z - \mu\left(\frac{\dot{N}_0}{N_0}\dot{E}_z + \frac{k_0^2}{\alpha^2}\Omega p^2E_z\right) = 0 \quad (5)$$

where the coefficient μ has been added to separate terms that cause difficulty in the solution, where $k_0=\omega/c$, and the dot indicates ∂/∂_z. Assuming the bracketed term to be small, a condition that requires the plasma to have only a slight gradient with respect to z, it will be shown that this equation can be solved approximately analytically using the technique of variation of parameters [10].

Following the procedure for variation of parameters, one first solves (5) with $\mu=0$ or

$$\ddot{E}_z + \frac{k_0^2}{\alpha^2}E_z = 0 \quad (6)$$

whose solution is

$$E_z = E_{z0}\cos\left(\frac{k_0}{\alpha}z + \theta\right)$$

$$\dot{E}_z = -\frac{k_0}{\alpha}E_{z0}\sin\left(\frac{k_0}{\alpha}z + \theta\right). \quad (7)$$

Let $\psi=(k_0/\alpha)z+\theta$ and let E_{z0} and θ be functions of z. One therefore obtains

$$\dot{E}_{z0}\cos\psi - E_{z0}\dot{\theta}\sin\psi = 0$$

$$\dot{E}_{z0}\sin\psi + E_{z0}\dot{\theta}\cos\psi = \frac{\dot{N}_0}{N_0}E_{z0}\sin\psi - \Omega p^2\frac{k_0}{\alpha}E_{z0}\cos\psi \quad (8)$$

from which one can write

$$\dot{\theta} = \frac{\dot{N}_0}{N_0}\cos\psi\sin\psi - \Omega p^2\frac{k_0}{\alpha}\cos^2\psi$$

$$\frac{\dot{E}_{z0}}{E_{z0}} = \frac{\dot{N}_0}{N_0}\sin^2\psi - \Omega p^2\frac{k_0}{\alpha}\cos\psi\sin\psi. \quad (9)$$

With a known variation of number density N_0 and a known temperature α, (9) can be solved for θ and E_{z0}, the results substituted into (7), and the longitudinal field will then be known for any z explicitly.

If these were not exactly known and since we have previously restricted the bracketed term to be small, i.e., nonrapidly varying, we can determine the average values of $\dot{\theta}$ and $\dot{E}_{z0}/E_{z0}$ to be

$$[\dot{\theta}]_{\text{av}} = -\frac{\Omega p^2}{2}\frac{k_0}{\alpha} \quad (10)$$

$$\left[\frac{\dot{E}_{z0}}{E_{z0}}\right]_{\text{av}} = \frac{\dot{N}_0}{2N_0}$$

which can then be integrated to yield

$$\theta = -\frac{\Omega p^2}{2}\frac{k_0}{\alpha}z + c_1$$

$$E_{z0} = C_2\epsilon^{(\dot{N}_0/2N_0)z}. \quad (11)$$

Substituting (11) into (7), one finally obtains for the longitudinal field

$$E_z = C_2\epsilon^{(\dot{N}_0/2N_0)z}\cdot\cos\left\{\frac{k_0}{\alpha}\left(1 - \frac{\Omega p^2}{2}\right)z + C_1\right\}. \quad (12)$$

The constants C_1 and C_2 can be determined from initial conditions. One immediately recognizes that the term $(1-\Omega p^2/2)$ is the expansion of the underdense relative dielectric constant $\sqrt{\epsilon_{p1}/\epsilon_0}$, an expected result.

One observes that this longitudinal wave propagates with a propagation constant that is temperature dependent and decays (or grows) as a function of the inhomogeneity of the plasma. This decay (growth) is in addition to that predicted by Landau and calculated by Fried and Gould [11]. Although their calculations are for a homogeneous plasma, it might be interesting to compare the Landau damping in the homogeneous case with that expected in the inhomogeneous case considered in this work. It must be pointed out though, that recent calculations by Harker indicate that Landau damping may not occur in certain cases where the distribution is parabolic [12].

However in the homogeneous case, we would expect Landau damping and from Fried and Gould, we can estimate the e-folding distance. With the ratio of $v_{\text{phase}}/v_{\text{Thermal}}=2$, they calculate the damping ratio $(\gamma/k)/v_{\text{Thermal}}\sim 1/4$. With the phase velocity defined as ω/k, one can show that the e-folding distance is

$$D = \frac{4}{\pi}\cdot\frac{v_{\text{phase}}}{f}. \quad (13)$$

In a typical mercury vapor discharge plasma, one can estimate this distance to be $4/\pi$ m at a frequency of 2 mc/s. From the measurements of Agdur et al. [6], one can estimate the e-folding distance from our consideration to be 0.43 meters for the case of a discharge current of 1 ampere.[1] At very high frequencies, this effect would most probably be negated by the Landau damping. In addition, one must be cognizant of the collisional damping which we have neglected in this calculation.

Since this longitudinal wave possesses an amplitude term that is a function of the density and a phase term that is a function of the temperature and density, one is led to postulate an application of this wave as diagnostics with microwaves to determine

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[1] One notes that there is a drift velocity associated with this tube. This would indicate that we should consider this in deriving the wave equation (2). However, an estimate of the phase velocity from (4) indicates it is much larger than the measured drift velocity [4]; we therefore neglect it.

the number density and temperature of a hot inhomogeneous plasma. This could be done by measuring the phase and the amplitude of the wave as a function of z.

Appendix

Derivation of the Wave Equation in a Hot Inhomogeneous Stationary Plasma

Assume all quantities are of the form A_0+A_1 where A_0 indicates time-independent quantities and A_1 indicates time-varying quantities. Further assume that the ions of density N_0 are smeared out and stationary to form a neutralizing background. The instantaneous charge density

$$\rho = -(N_0 + N_1)e + N_0 e = -N_1 e \quad (14)$$

and current

$$\bar{J} = -(N_0 + N_1)e(\bar{v}_1) = -N_0 e \bar{v}_1 \quad (15)$$

together with Maxwell's equations

$$\frac{\partial \epsilon_0 \bar{E}}{\partial t} + \bar{J} = \text{curl}\,\frac{\bar{B}}{\mu_0} \quad (16)$$

$$\frac{\partial \bar{B}}{\partial t} + \text{curl}\,\bar{E} = 0 \quad (17)$$

$$\text{div}\,\epsilon_0 \bar{E} = \rho \quad (18)$$

lead to expressions (with assumed time dependence $\epsilon^{i\omega t}$)

$$+N_0 e \bar{v}_1 = \frac{i}{\omega\mu_0}\left(\frac{\omega^2}{c^2}\bar{E}_1 - \text{curl curl}\,\bar{E}_1\right) \quad (19)$$

$$N_1 = -\frac{\epsilon_0}{e}\,\text{div}\,\bar{E}_1. \quad (20)$$

The linearized equation of motion (1) yields two terms

$$-eN_0\bar{E}_0 - \text{grad}\,P_0 = 0 \quad (21)$$

$$i\omega m N_0 \bar{v}_1 = -eN_1\bar{E}_0 - eN_0\bar{E}_1 - \text{grad}\,P_1. \quad (22)$$

Combining (19) to (22) together with an assumed equation of state $p=\gamma NkT$, one obtains the wave equation (2).

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References

[1] B. Keržar and P. Weissglas, "Plasma microwave interaction," *J. Appl. Phys.*, vol. 36, p. 2479, August 1965.
[2] B. O'Brien, R. W. Gould, and J. Parker, "Longitudinal wave-propagation modes along a positive column," *Phys. Rev. Lett.*, vol. 14, p. 630, April 19, 1965.
[3] R. A. Stern, "Harmonic Generation and Frequency Mixing at Plasma Resonance," *Phys. Rev. Lett.* vol. 14, p. 538, April 5. 1965.
[4] B. Keržar, K. Abrahamsen, and P. Weissglas, "Nonlinear plasma wave interaction in an arc discharge," *Appl. Phys. Lett.*, vol. 7, p. 155, September 15, 1965.
[5] B. Keržar and P. Weissglas, "Determination of electron drift velocity in a gaseous discharge from Doppler shifts of plasma waves," *Electronics Lett.*, vol. 1, p. 43, April 1965.
[6] B. Agdur, B. Keržar, and T. Nygren, "Variation of the electron density along a plasma column," *Phys. Rev. Lett.*, vol. 10, p. 467, June 1, 1963.
[7] I. Alexeff and W. D. Jones, "Collisionless ion-wave propagation and the determination of the compression coefficient of plasma electrons," *Phys. Rev. Lett.*, vol. 15, p. 286, August 16, 1965.
[8] B. Agdur and P. Weissglas, "Waves in hot, anisotropic electron plasmas," *Ericsson Tech.*, p. 267, no. 2, 1964.
[9] J. C. Nickel, J. V. Parker, and R. W. Gould, "Resonance oscillations in a hot nonuniform plasma column," *Phys. Rev. Lett.*, vol. 11, p. 183, September 1, 1963.
[10] W. J. Cunningham, *Introduction to Nonlinear Analysis*. New York: McGraw-Hill, 1958, pp. 135–140.
[11] B. D. Fried and R. W. Gould, "Longitudinal ion oscillations in a hot plasma," *Phys. of Fluids*, vol. 4, p. 139, January 1961. (See Fig. 1.)
[12] K. J. Harker, "Longitudinal oscillations in unbounded one-dimensional nonuniform plasmas," *Phys. of Fluids*, vol. 10, p. 1846, October 1965.

Comments on Excitation of Spin Waves by Wire Arrays

Messrs. LaRosa and Vasile[1]

Kaufman and Soohoo[2] have suggested that spin waves involving exchange forces can be excited by means of a fine wire at the end of a YIG crystal. It would be very desirable to do this, for then the entire crystal might be kept at a low enough dc field to avoid coupling to acoustic waves. We have made a detailed analysis of an array of flat wires immersed in an infinite YIG medium and also printed on the air-YIG interface of a semi-infinite YIG medium. The dc magnetic field has been taken both perpendicular to the plane of the array and parallel to the array conductors.

One idea which runs through the literature on the subject is that the rapid decay of the exciting field in the desired propagation direction should enable some net coupling to be obtained to the very short wavelength (10^{-5} cm) exchange spin waves. Accordingly, the currents in the adjacent wires were assumed to be oppositely directed. The assumption of an infinite array enabled the boundary value problem to be solved in rectangular coordinates. The magnetic field and the RF magnetization component could be expressed in terms of Fourier sine and cosine series as functions of the coordinate along the array (perpendicular to the conductors). No variation along the conductors was assumed. The variation perpendicular to the array has several propagation constant values given by a dispersion relation for each orientation of the dc field.

For the dc field perpendicular to the array, four values of wave vector k were found.

1) Low-k (electromagnetic). Decaying in propagation direction.
2) Medium-k propagating. Similar to magnetostatic waves (group velocity opposite to phase velocity).
3) High-k propagating. Involving exchange forces, obeying $\omega/\gamma = H_i + H_{ex} a^2k^2$.
4) High-k, nonpropagating. Circular polarization sense opposite to the high-k propagating. Obeying dispersion relation $-\omega/\gamma = H_i + H_{ex} a^2k^2$ (k^2 negative).

It was found that for all conductor widths and spacings, only low-k waves were excited. In fact, the ability of the low-k waves to satisfy the boundary conditions is enhanced by decreasing the width and spacing of the conductors. The plausibility argument for this is that the electromagnetic field of the conductors without any YIG obeys very closely Laplace's equation, i.e., $k\approx 0$. Therefore, there is no variation in the exciting field which tends to displace the spins against the exchange forces. Very close conductor spacing creates a fast decay perpendicular to the array and an equally fast periodic variation along the array. The second derivatives in the two directions are equal and opposite and the Laplacian is zero.

The situation is slightly different with the dc field parallel to the wires. The closeness of the conductors drops out of the dispersion relation. Three values of k are permitted.

1) Low-k (electromagnetic) becomes high-k nonpropagating as the field is increased.
2) High-k propagating becomes low-k as the field is increased.
3) High-k nonpropagating remains about the same.

There is a restricted range of field in which 1) and 2) become comparable, so that true spin waves involving exchange forces are excited. However, good excitation of propagating spin waves occurs only for spin wavelengths comparable to or greater than elastic wavelengths. Also, this region is very narrow band.

The transmission line analog used by Kaufman and Soohoo[3] assumes single mode propagation and reduces a two- or three-dimensional problem to a one-dimensional problem. The actual magnetic field of the conductor is replaced by sources distributed along the transmission line. This distributed source analog is based on Schlömann,[4] equation (10), which we believe to be a misinterpretation, as follows.

The equations of motion give the susceptibility tensor which relates the RF magnetization to the RF magnetic field in a gyrotropic medium. This tensor expresses the effects of forces imposed by the local magnetic field and the exchange forces on the spin dipole moments. These forces cause precession at an amplitude and rate consistent with the magnetogyric ratio.

When the susceptibility tensor is inserted in Maxwell's equations, a set of homogeneous equations can be obtained for either the magnetization or the magnetic field. The dispersion relation is obtained by setting the determinant equal to zero.

There are no source terms involved in the interior of the region. Amplitudes are determined by matching solutions at boundaries.

[1] Manuscript received November 8, 1965.

[2] I. Kaufman and R. F. Soohoo, "Properties and excitation of spin waves—A new microwave time delay medium," presented at 1964 PTGMTT International Symposium.

[3] I. Kaufman and R. F. Soohoo, "Magnetic waves for microwave time delay—Some observations and results," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-13, pp. 458–467, July 1965.

[4] E. Schlömann, "Generation of spin waves in nonuniform magnetic fields, I. Conversion of electromagnetic power into spin-wave power and vice versa," *J. Appl. Phys.*, vol. 35, p. 159–166, January 1964.